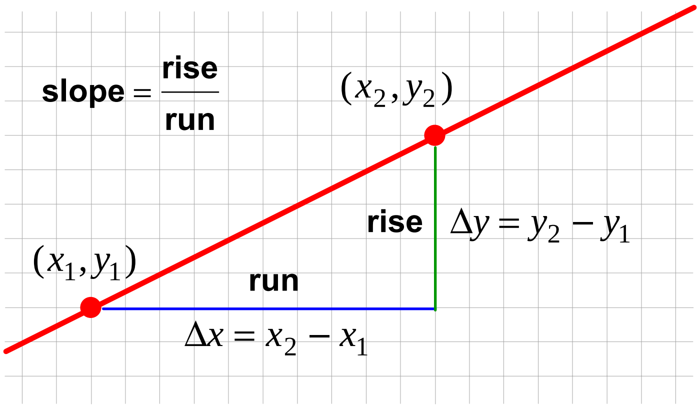
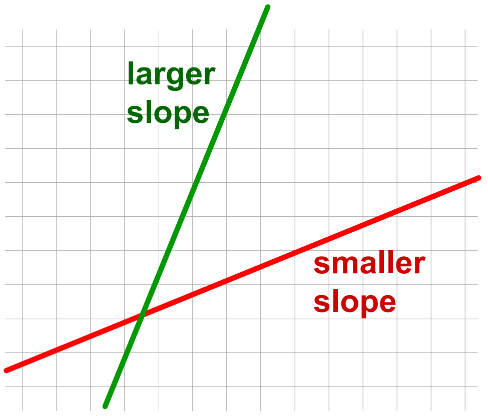
Lesson 3.1

Slopes of Curves; Derivatives

Slope of a line (recap) The slope of a line measures its "steepness": for any two points (x_1, y_1) and (x_2, y_2) on the line, the slope is the ratio of the "rise" $\Delta y = y_2 - y_1$ to the "run" $\Delta x = x_2 - x_1$, i.e.

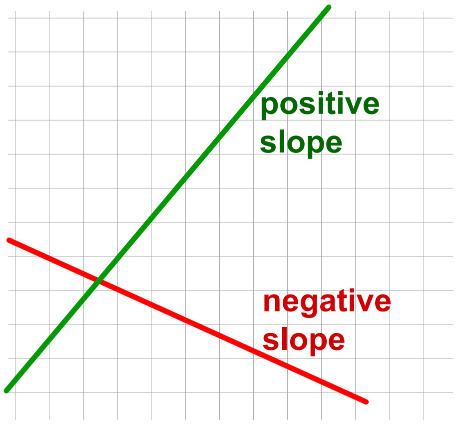
$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x}.$$





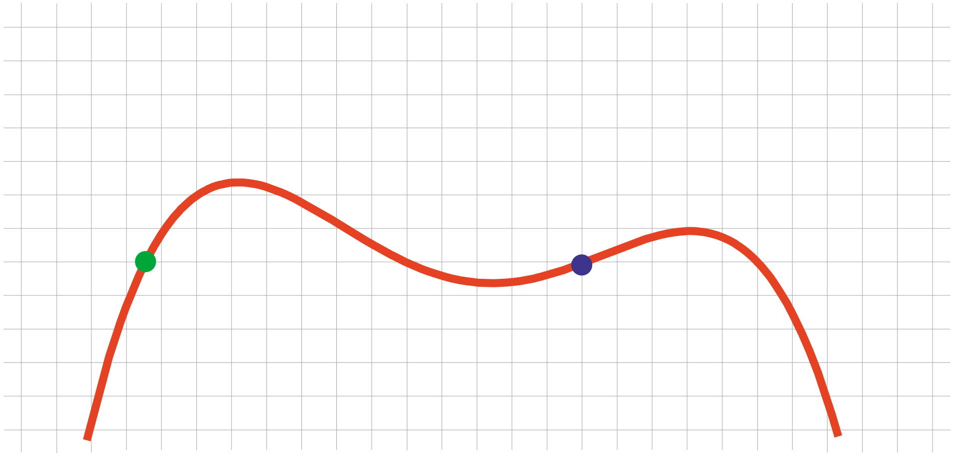
Steep lines have large slopes (large rise relative to run), flatter lines have smaller slopes (small rise relative to run).

Increasing lines (going up towards the right) have positive slopes (same sign rise and run). Decreasing lines (going down towards the right) have negative slopes (opposite sign rise and run).



The slope of a line doesn't depend on the pair of points on the line used to calculate it; all pairs of points on the same line will give the same slope.

Local slope of a curve For curves that aren't lines, a single overall slope is not a very useful measure of steepness. Intuitively, the steepness of a typical curve is different at different places along the curve - if the curve in the diagram represented a mountain range, for example, we would say it is steeper at the left-hand marked point than at the right-hand one.



The quantity $\Delta y / \Delta x$ is different for different pairs of points in the curve, and gives at best an "average slope" of the curve between the points - for the two marked points in the diagram, for example, it is 0 (since $\Delta y = 0$), and so gives no information about the hump and valley between the points.

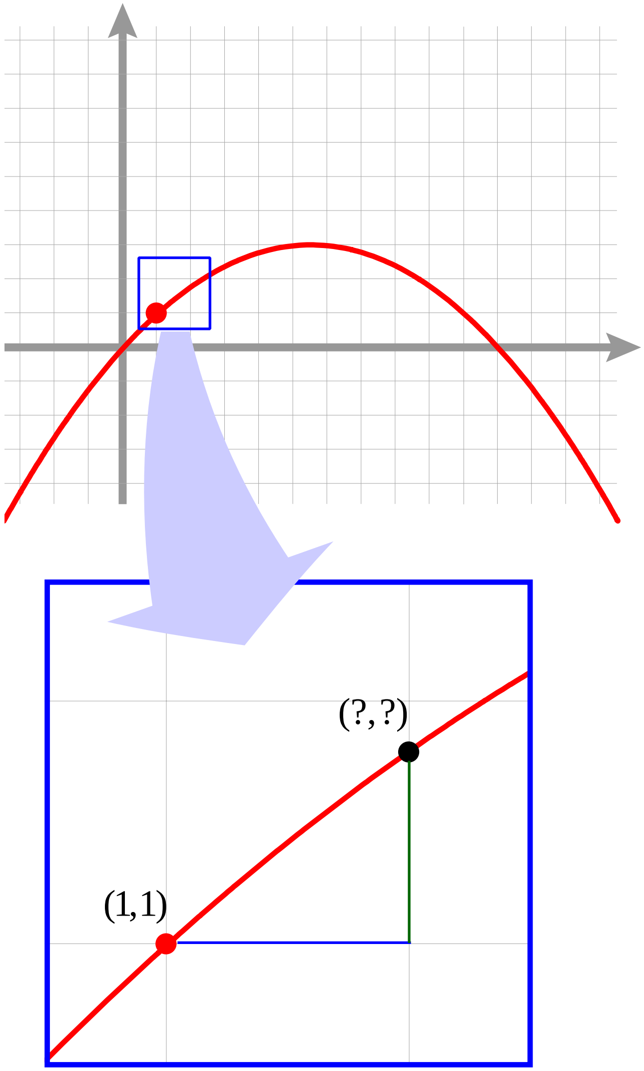
We need to **modify the definition of slope** to reflect the variable steepness of a curve. Let's investigate by looking at a simple "one-hump" curve

$$y = \frac{1}{10}x(11 - x)$$

near a typical point on it, say $(1,1)$.

Plot the curve and the point, zoom in several times on the point, and notice what happens as you zoom: the closer you get to the point, the more the visible part of the curve resembles a line - it "straightens out".

Read off from your zoomed graph the coordinates of another point on the curve and use it with the point $(1,1)$ to calculate $\Delta y / \Delta x$. You should get a number close to 0.9. "Locally" – in a small region near the point $(1,1)$ – the curve resembles a straight line with slope approximately 0.9.



Let's **work this out algebraically**. Suppose the nearby point has x-coordinate near 1, so its x-coordinate is $x = 1 + h$ for some small $h \neq 0$. Its y-coordinate is then

$$y = \tfrac{1}{10}(1+h)[11-(1+h)] = \tfrac{1}{10}(10+9h-h^2)$$

so

$$\begin{aligned} \frac{\Delta y}{\Delta x} &= \frac{\tfrac{1}{10}\left(10+9h-h^2\right)-1}{(1+h)-1} \\ &= \frac{9-h}{10} \\ &= 0.9-(0.1)h \end{aligned}$$

Since h is small, this is close to 0.9, as expected.

Now zoom in closer, pick another nearby point and calculate $\Delta y / \Delta x$ again. The visible part of the curve appears straighter, and the nearby point must be chosen closer to $(1,1)$ than before, so h will be smaller than before and $\Delta y / \Delta x$ will be closer to 0.9.

Suppose you repeat this process over and over. In effect, what you are then doing is a limiting process: you are taking the limit of

$$\frac{\Delta y}{\Delta x} = 0.9-(0.1)h$$

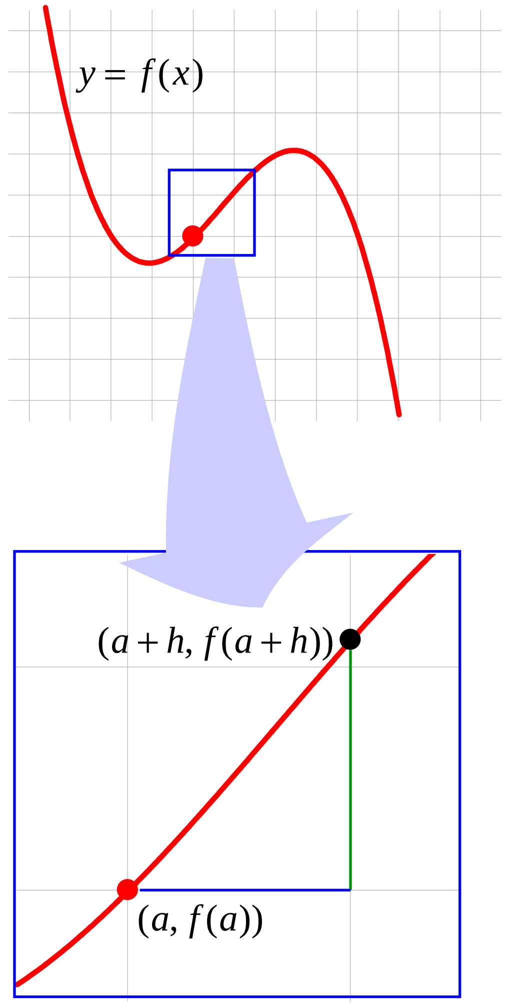
as h approaches 0. The curve "straightens out" in the process, so the result 0.9 is a good candidate for the slope of the curve near $(1,1)$.

Let's **formalize this process into a definition.** Suppose we have the graph of a general function $y = f(x)$ and a general point $(a, f(a))$ on it.

Zoom in on $(a, f(a))$ and calculate $\Delta y / \Delta x$ from the given point $(a, f(a))$ and a nearby point on the curve. This point has x-coordinate $x = a + h$ for some small value h and y-coordinate $y = f(a+h)$, so

$$\frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h}.$$

The closer the zoomed part of the curve resembles a line, the better candidate the quantity $\Delta y / \Delta x$ is for the slope of the curve near the point $(a, f(a))$.



Repeat the process: zoom and calculate. At each step, the nearby point $(a+h, f(a+h))$ must be closer to $(a, f(a))$, so the value of h must be closer to 0. If the visible part of the curve becomes straighter as you zoom, the number $\Delta y / \Delta x$ becomes a better description the slope of the curve near $(a, f(a))$. It then makes sense to define the slope of the curve *at* $(a, f(a))$ to be the limit of $\Delta y / \Delta x$ as $h \to 0$.

Slope of a curve The ***slope of a curve*** $y = f(x)$ at the point $(a, f(a))$ on it is defined to be the number

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.

Quick Question Find the slope of the curve $y = \frac{1}{10}x(11 - x)$ at the point $(6, 3)$ by zooming and then check your answer algebraically.

Example 3.1.1 Find the exact slope of the curve $y=\sqrt{x}$ at the point $(4,2)$.

Solution Here $a=4$ and $f(x)=\sqrt{x}$, so the slope is

$$\begin{aligned}
\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} &= \lim_{h\to 0}\frac{\sqrt{4+h}-\sqrt{4}}{h} \\
&= \lim_{h\to 0}\frac{\sqrt{4+h}-2}{h}\times\frac{\sqrt{4+h}+2}{\sqrt{4+h}+2} \\
&= \lim_{h\to 0}\frac{(4+h)-4}{h(\sqrt{4+h}+2)} \\
&= \lim_{h\to 0}\frac{1}{\sqrt{4+h}+2} \\
&= \frac{1}{\sqrt{4}+2} \\
&= \tfrac{1}{4}
\end{aligned}$$

Check this answer by zooming in on the graph of $y=\sqrt{x}$ near $(4,2)$. ■

Problem 3.1.1 Find the slope of the curve $y=\frac{1}{x}$ at $(2,\frac{1}{2})$ algebraically and then check your answer by zooming in on the graph.

Derivatives The special limit used to find slopes of curves occurs in many other contexts, and so has a name and a notation.

> **Derivative at a point** For any function $y = f(x)$, the number
>
> $$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$
>
> if it exists, is called the ***derivative*** of the function f at the value $x = a$.

Example 3.1.2 Calculate the derivative of $f(x) = (x+3)^2$ at $x = 0$.

Solution

$$\begin{aligned} f'(0) &= \lim_{h \to o} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h \to o} \frac{(0+h+3)^2 - (0+3)^2}{h} \\ &= \lim_{h \to o} \frac{(h^2+6h+9) - (9)}{h} \\ &= \lim_{h \to o} (h+6) \\ &= 6 \quad \blacksquare \end{aligned}$$

Problem 3.1.2 Use the definition to find the derivative of $f(x) = \sqrt{x}(x-2)$ at $x = 2$.

Quick Question Show that, at $x = 0$, the derivative of $\sin x$ is 1 and the derivative of $\cos x$ is 0.

Quick Question Identify "by inspection" a function f and a number a such that

$$f'(a) = \lim_{h \to 0} \frac{(2+h)e^{(2+h)} - 2e^2}{h}.$$

Quick Question Without calculation or graphing, give the derivative of $f(x) = 37x + 41$ at $x = 11$.

Example 3.1.3 Use the definition to calculate the derivative of $f(x)=\frac{x^2}{x+1}$ at $x=2$.

Solution

$$f'(2)=\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$

$$=\lim_{h\to 0}\frac{\frac{(2+h)^2}{(2+h)+1}-\frac{2^2}{2+1}}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left\{\frac{(2+h)^2}{3+h}-\frac{4}{3}\right\}$$

$$=\lim_{h\to 0}\frac{1}{h}\left\{\frac{3(2+h)^2-(3+h)4}{(3+h)(3)}\right\}$$

$$=\lim_{h\to 0}\frac{1}{h}\left\{\frac{\left(12+12h+3h^2\right)-(12+4h)}{(3+h)(3)}\right\}$$

$$=\lim_{h\to 0}\frac{1}{h}\left\{\frac{8h+3h^2}{3(3+h)}\right\}$$

$$=\lim_{h\to 0}\frac{8+3h}{3(3+h)}$$

$$=\frac{8}{3(3)}=\frac{8}{9}$$ ■

Problem 3.1.3 Use the definition of a derivative to calculate $f'(3)$ for $f(x)=\frac{1}{\sqrt{x+1}}$ at $x=3$.

Derivative calculations with your CAS. To calculate derivatives of more complicated functions, you can often use your CAS to do the necessary work. To set this up so you can reuse your work, first define the ***difference quotient***

$$\text{difference quotient} = \frac{f(a+h)-f(a)}{h}.$$

Then, for each derivative you want to find, define the function f and the value of a, and substitute both into the difference quotient. Use your CAS to simplify the difference quotient (don't multiply out the denominator - you need a factor h in the denominator to cancel a factor h in the simplified numerator). Finally, take the limit as h approaches 0.

Problem 3.1.4 Use your CAS to find the slope of the curve

$$y = \frac{x^2-5}{x^2+1}$$

at the point $(4, \frac{11}{17})$.

Solutions to the Problems

Problem 3.1.1 Find the slope of the curve $y = \frac{1}{x}$ at $(2, \tfrac{1}{2})$ algebraically and then check your answer by zooming in on the graph.

Solution For $f(x) = \frac{1}{x}$, the slope of the curve at $(2, \tfrac{1}{2})$ is

$$\lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \left\{ \frac{1}{2+h} - \frac{1}{2} \right\}$$

$$= \lim_{h \to 0} \frac{1}{h} \left\{ \frac{2 - (2+h)}{2(2+h)} \right\}$$

$$= \lim_{h \to 0} \frac{-1}{2(2+h)}$$

$$= \frac{-1}{2(2)}$$

$$= -\tfrac{1}{4}$$

Problem 3.1.2 Use the definition to find the derivative of $f(x)=\sqrt{x}(x-2)$ at $x=2$.

Solution

$$f'(2)=\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$

$$=\lim_{h\to 0}\frac{\sqrt{2+h}(2+h-2)-\sqrt{2}(2-2)}{h}$$

$$=\lim_{h\to 0}\left(\sqrt{2+h}\right)$$

$$=\sqrt{2}$$

Problem 3.1.3 Use the definition to calculate the derivative of the function $f(x)=\frac{1}{\sqrt{x+1}}$ at $x=3$.

Solution

$$\begin{aligned}
f'(3) &= \lim_{h\to 0}\frac{f(3+h)-f(3)}{h} \\
&= \lim_{h\to 0}\frac{\frac{1}{\sqrt{(3+h)+1}}-\frac{1}{\sqrt{3+1}}}{h} \\
&= \lim_{h\to 0}\frac{1}{h}\left\{\frac{1}{\sqrt{4+h}}-\frac{1}{2}\right\} \\
&= \lim_{h\to 0}\frac{1}{h}\left\{\frac{2-\sqrt{4+h}}{2\sqrt{4+h}}\right\} \\
&= \lim_{h\to 0}\frac{1}{h}\left\{\frac{2-\sqrt{4+h}}{2\sqrt{4+h}}\times\frac{2+\sqrt{4+h}}{2+\sqrt{4+h}}\right\} \\
&= \lim_{h\to 0}\frac{1}{h}\left\{\frac{4-(4+h)}{2\sqrt{4+h}\left(2+\sqrt{4+h}\right)}\right\} \\
&= \lim_{h\to 0}\frac{-1}{2\sqrt{4+h}\left(2+\sqrt{4+h}\right)} \\
&= \frac{-1}{2\sqrt{4}\left(2+\sqrt{4}\right)} \\
&= -\tfrac{1}{16}
\end{aligned}$$

Problem 3.1.4 Use your CAS to find the slope of the curve

$$y=\frac{x^2-5}{x^2+1}$$

at the point $(4,\frac{11}{17})$.

Solution The slope is $f'(4)$, where $f(x)=\dfrac{x^2-5}{x^2+1}$.

$$\begin{aligned}
\text{difference quotient} &= \frac{f(4+h)-f(4)}{h}\\
&=\frac{1}{h}\left\{\frac{(4+h)^2-5}{(4+h)^2+1}-\frac{4^2-5}{4^2+1}\right\}\\
&=\frac{1}{h}\left\{\frac{16+8h+h^2-5}{16+8h+h^2+1}-\frac{11}{17}\right\}\\
&=\frac{1}{h}\left\{\frac{11+8h+h^2}{17+8h+h^2}-\frac{11}{17}\right\}\\
&=\frac{1}{h}\left\{\frac{17\left(11+8h+h^2\right)-11\left(17+8h+h^2\right)}{17\left(17+8h+h^2\right)}\right\}\\
&=\frac{1}{h}\left\{\frac{48h+6h^2}{17\left(17+8h+h^2\right)}\right\}\\
&=\frac{48+6h}{17\left(17+8h+h^2\right)}
\end{aligned}$$

The derivative is then

$$\begin{aligned}
f'(4) &= \lim_{h\to 0}\left\{\text{difference quotient}\right\}\\
&=\lim_{h\to 0}\frac{48+6h}{17\left(17+8h+h^2\right)}\\
&=\tfrac{48}{17\times 17}\\
&=\tfrac{48}{289}
\end{aligned}$$

Your calculations may appear different, depending on how you use your CAS to simplify the difference quotient.