

Experimental Lesson for an Onscreen Calculus Course: the Definition of the Derivative

This file is a demonstration of a prototypical onscreen lesson implementing several instructional design ideas. The main intent was to experiment with a design principle of John Sweller (*[Instructional Design in Technical Areas](#)*, ACER Press, 1999) on using audio to increase working memory capacity and hence improve learning. (A [table describing Sweller's principles](#), interpreted somewhat in the context of this lesson, may be found at the end of this file; the reader is strongly encouraged to look at the book for details and the research upon which they are based.) Other instructional design ideas are described by annotations to the pages which follow, and at the end in [Pedagogy Notes](#). Details of the technology used can be found in [Technology Notes](#). For comments or questions, please email June Lester <jalester@cecm.sfu.ca>

Instructions for the Demo.

To hear the voiceovers, click on the speaker icons (try this one).



To read the annotations, double click on the comment icons (try it here). These aren't part of the lesson; they have been added for this demo only.



Acrobat settings for optimal viewing (optional): Since the pages are different lengths, it's best if the view you choose (under the View menu) is "actual size". Also choose the page layout "single page" from the View menu to see one page at a time. In order to see useful page numbers in the status bar (at the bottom of the window), check the "use logical page numbers" box in the Preferences > General... popup window (look under the right arrow at the top right of the Acrobat Reader window). Also check the "use page cache" box in this same window to speed up loading of individual pages (if you have sufficient RAM).

Lesson 3.1

Slopes of Curves; Derivatives

What this lesson is about:

This lesson introduces the most fundamental idea of the course: the derivative. As you work through the lesson, you will learn how to

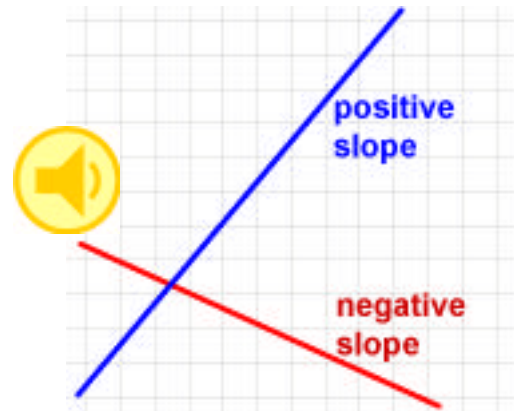
- find the slope of a curve at a given point on it
- define the derivative of a function at a point
- use the definition to calculate derivatives.

What you need to know to understand this lesson:

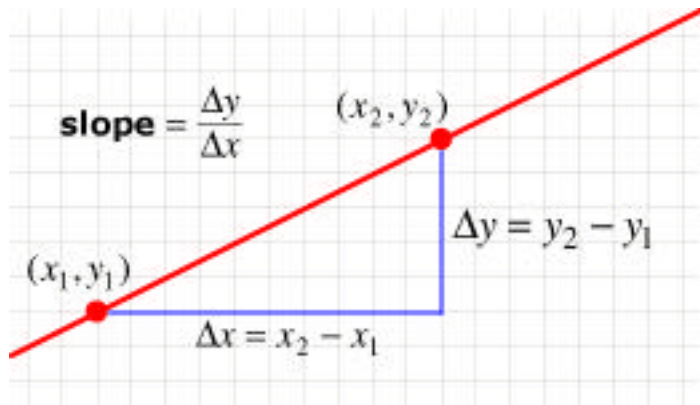
- what the slope of a line is and how to calculate it
- how to calculate limits

This lesson should take you P - Q hours to complete.

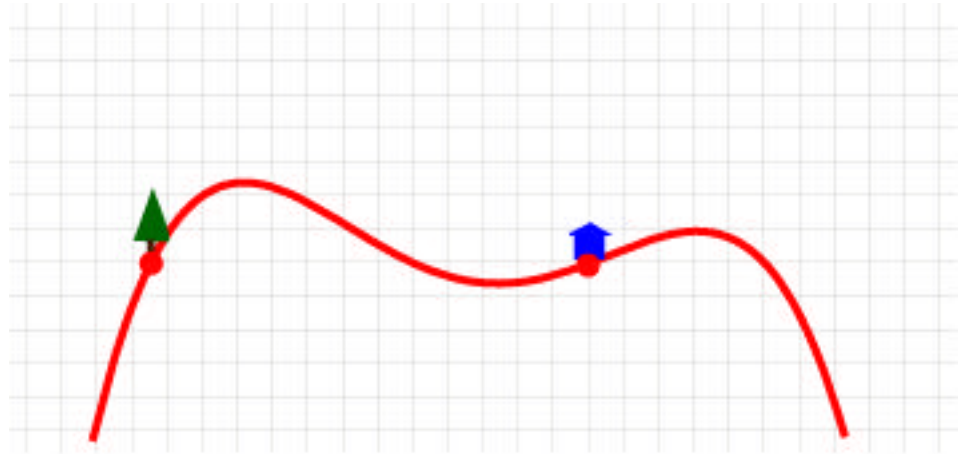
What you should know about slopes of lines (review).



Slopes of lines are difference quotients.




What do we mean by the slope of a curve?

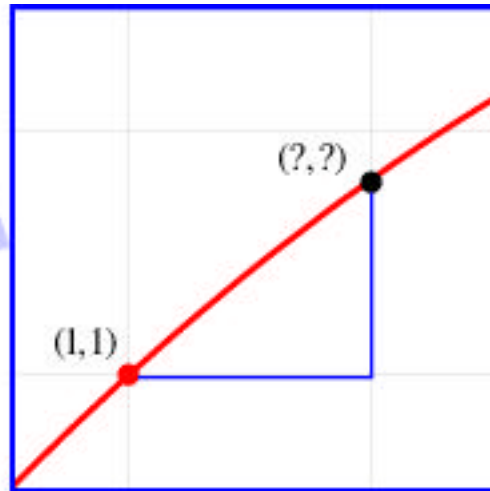
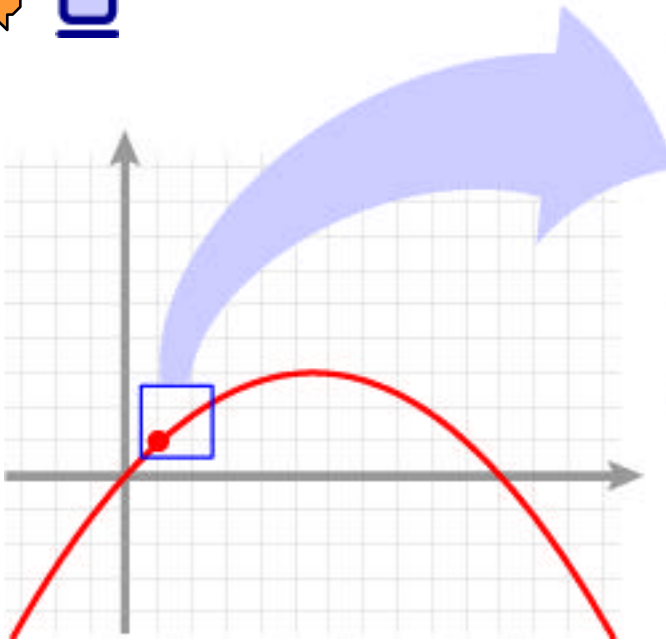


Explore a sample curve close up.



curve: $y = \frac{1}{10}x(11-x)$ 

point: (1,1)



The shape of the zoomed curve ...



When you've finished calculating ...



Do this same exploration algebraically.

curve: $y = \frac{1}{10}x(11-x)$

point: $(x_1, y_1) = (1, 1)$

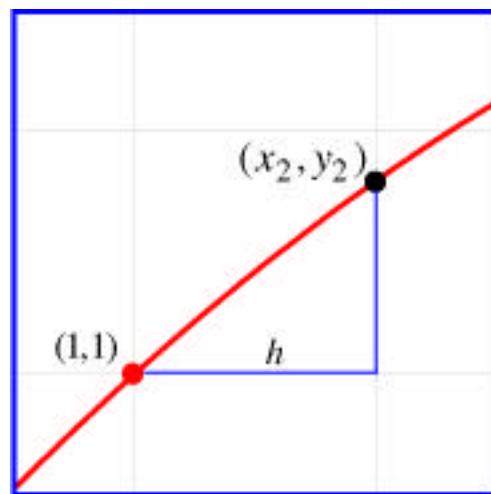
nearby point:



$$x_2 = 1 + h$$



$$\begin{aligned}y_2 &= \frac{1}{10}(1+h)[11-(1+h)] \\&= \frac{1}{10}(1+h)(10-h) \\&= \frac{1}{10}(10+9h-h^2) \\&= 1 + (0.9)h - (0.1)h^2\end{aligned}$$



difference quotient:

$$\begin{aligned}\frac{y}{x} &= \frac{y_2 - y_1}{x_2 - x_1} \\&= \frac{[1 + (0.9)h - (0.1)h^2] - 1}{(1+h) - 1} \\&= \frac{(0.9)h - (0.1)h^2}{h} \\&= 0.9 - (0.1)h\end{aligned}$$

What this difference quotient tells you:



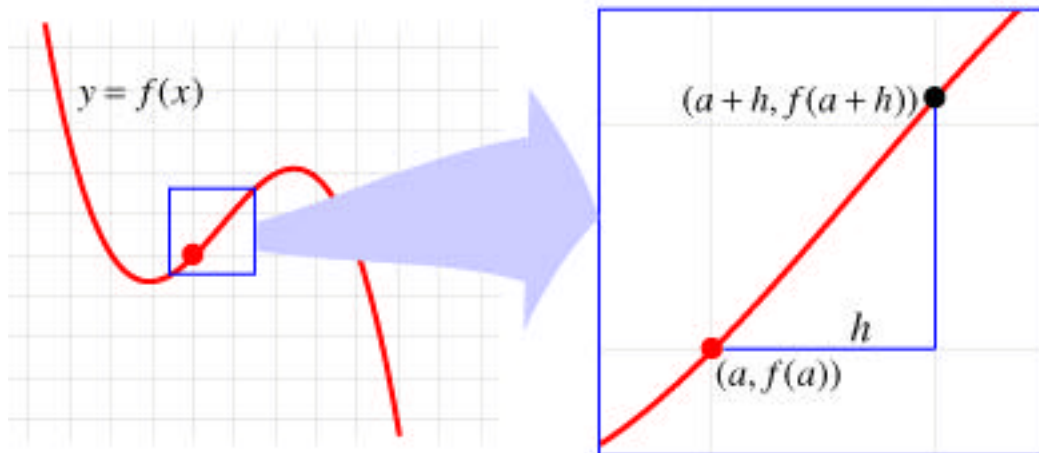
Suppose you do this over and over ...



Now make what you just did more general.



- **general curve** $y = f(x)$
- **general point on it** $(a, f(a))$
- **zoom**



- **nearby point** $(a + h, f(a + h))$
- **difference quotient**

$$\begin{aligned} \frac{y}{x} &= \frac{f(a + h) - f(a)}{(a + h) - a} \\ &= \frac{f(a + h) - f(a)}{h} \end{aligned}$$



Definition

The **slope of a curve** $y = f(x)$ at the point $(a, f(a))$ on it is the limit of a difference quotient:

$$slope = \lim_{x \rightarrow a} \frac{y}{x} = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}$$

if this limit exists.



✓ Find the slope of the curve $y = \frac{1}{10}x(11 - x)$ at the point $(6, 3)$ on it by zooming and then check your answer algebraically.

Defining derivatives.



Definition

For any function $y = f(x)$, the limit

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

if it exists, is called the **derivative** of f at $x = a$.

✓ For the function $f(x) = \frac{1}{10}x(11-x)$, since the slope of $y = \frac{1}{10}x(11-x)$ at $(1,1)$ is 0.9, we have $f'(1) = 0.9$. What is $f'(6)$?



For $f(x) = 37x + 73$, find $f'(13)$ "by inspection"; that is, without graphing the function or doing a calculation.



Calculating Derivatives

To do these warmups, you'll need to remember the derivative formula

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$



Sample

Calculate the derivative of x^2 at $x = 1$.

Solution Here $a = 1$ and $f(x) = x^2$, so

$$\begin{aligned}
 f'(1) &= \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(1+h)^2 - 1^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{1} + 2h + h^2 - \cancel{1}}{h} \\
 &= \lim_{h \rightarrow 0} \{2 + h\} \\
 &= 2
 \end{aligned}$$



Put in the value of a .



Put in the function f .



Simplify and calculate the limit.

Done

Exercises



Calculate some of the following derivatives. You've warmed up enough once you can do these fairly quickly without having to look up the formula each time.

- a) the derivative of x^3 at $x = 2$
- b) $F'(5)$ for $F(t) = (t-5)\sqrt{t}$
- c) the derivative of $x^{5/3}$ at $x = 0$
- d) $f'(0)$ if $f(x) = \sin x$

- e) the derivative of $1 - 7s$ at $s = 5$
- f) $g'(0)$, given $g(x) = x^2 - 2x$
- g) the derivative of $\cos x$ at $x = 0$
- h) $s'(-1)$ when $s(x) = (x+1)e^x$

Answers to the Warmups



a) 0

b) $\sqrt{5}$

c) 0

d) 1

e) -7

f) -2

g) 0

h) e^{-1}



Problems for Lesson 3.1

Example 3.1.1

Calculate the derivative of $f(x) = \frac{1}{x+3}$ at $x = 2$.

Solution

$$\begin{aligned}
 f'(2) &= \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{1}{[2+h]+3} - \frac{1}{2+3} \right) \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{1}{5+h} - \frac{1}{5} \right) \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \frac{5 - (5+h)}{(5+h)5} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \frac{-h}{(5+h)5} \\
 &= \lim_{h \rightarrow 0} \frac{-1}{(5+h)5} \\
 &= -\frac{1}{25}
 \end{aligned}$$

If the function is a fraction, it's neater to write division by h as multiplication by $1/h$.

Generally, you don't want to multiply out the h in the denominator - you want a separate factor h available in the denominator to cancel out an h in the numerator.

Done



Problem 3.1.1

Calculate the derivative of $f(x) = \frac{1}{x} + 3$ at $x = 2$.

Example 3.1.2

For $f(x) = \frac{1}{\sqrt{x}}$, calculate $f'(9)$.

Solution

$$\begin{aligned} f'(9) &= \lim_{h \rightarrow 0} \frac{f(9+h) - f(9)}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{1}{\sqrt{9+h}} - \frac{1}{\sqrt{9}} \right) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \frac{3 - \sqrt{9+h}}{3\sqrt{9+h}} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \frac{3 - \sqrt{9+h}}{3\sqrt{9+h}} \times \frac{3 + \sqrt{9+h}}{3 + \sqrt{9+h}} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \frac{9 - (9+h)}{3\sqrt{9+h}(3 + \sqrt{9+h})} \\ &= \lim_{h \rightarrow 0} \frac{-1}{3\sqrt{9+h}(3 + \sqrt{9+h})} \\ &= \frac{-1}{3\sqrt{9}(3 + \sqrt{9})} \\ &= -\frac{1}{54} \end{aligned}$$

Rationalize the numerator.

Don't waste time multiplying out the denominator here.

Now simplify the denominator.

Done

Problem 3.1.2

For $f(x) = \sqrt{7-x}$, calculate $f'(3)$.

Example 3.1.3

Calculate the derivative of $f(x) = (x + 4)\tan\left(\frac{x}{3}\right)$ at $x = -4$.

Solution

$$\begin{aligned} f'(-4) &= \lim_{h \rightarrow 0} \frac{f(-4+h) - f(-4)}{h} \\ &= \lim_{h \rightarrow 0} \frac{([-4+h] + 4)\tan\left(\frac{[-4+h]}{3}\right) - (-4+4)\tan\left(\frac{[-4]}{3}\right)}{h} \\ &= \lim_{h \rightarrow 0} \frac{h \tan\left(\frac{[-4+h]}{3}\right)}{h} \\ &= \lim_{h \rightarrow 0} \tan\left(\frac{[-4+h]}{3}\right) \\ &= \tan\left(\frac{-4}{3}\right) \\ &= -\sqrt{3} \end{aligned}$$

Done

Problem 3.1.3

For $s(t) = e^t \sin t$, find $s'(t)$.

Solution to Problem 3.1.1

Calculate the derivative of $f(x) = \frac{1}{x} + 3$ at $x = 2$.

$$\begin{aligned} f'(2) &= \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{1}{h} + \frac{1}{2+h} - \frac{1}{2} - 3 + 3}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \frac{2 - (2+h)}{(2+h)^2} \\ &= \lim_{h \rightarrow 0} \frac{-1}{(2+h)^2} \\ &= -\frac{1}{4} \end{aligned}$$

Solution to Problem 3.1.2

For $f(x) = \sqrt{7-x}$, calculate $f'(3)$.

$$\begin{aligned}f'(3) &= \lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h} \\&= \lim_{h \rightarrow 0} \frac{\sqrt{7-(3+h)} - \sqrt{7-3}}{h} \\&= \lim_{h \rightarrow 0} \frac{\sqrt{4-h} - 2}{h} \\&= \lim_{h \rightarrow 0} \frac{\sqrt{4-h} - 2}{h} \times \frac{\sqrt{4-h} + 2}{\sqrt{4-h} + 2} \quad \leftarrow \text{Rationalize the numerator.} \\&= \lim_{h \rightarrow 0} \frac{(4-h) - 4}{h(\sqrt{4-h} + 2)} \\&= \lim_{h \rightarrow 0} \frac{-1}{\sqrt{4-h} + 2} \\&= \frac{-1}{\sqrt{4} + 2} \\&= -\frac{1}{4}\end{aligned}$$

Solution to Problem 3.1.3

For $s(t) = e^t \sin t$, find $s'(0)$.

$$s'(0) = \lim_{h \rightarrow 0} \frac{s(0+h) - s(0)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{e^{+h} \sin(0+h) - e^0 \sin(0)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{-e^{+h} \sin(h)}{h}$$

since $\sin(0+h) = -\sin(h)$
and $\sin(0) = 0$

$$= - \lim_{h \rightarrow 0} e^{+h} \times \lim_{h \rightarrow 0} \frac{\sin(h)}{h}$$

$$= -e \times 1$$

since $\lim_{h \rightarrow 0} \frac{\sin(h)}{h} = 1$

$$= -e$$

Pedagogy Notes: the Lesson Format


The lesson has three basic teaching/learning goals, addressed by different segments of the lesson.

1. **Understanding**, addressed by a segment of explanation. In this particular lesson, we would like the student to understand *on an intuitive level* what a derivative is (the slope of a curve looked at close up) and that the formula for a derivative comes from this idea.
2. **Automation**, addressed by the “warmups” segment following the explanation. The warmups are “baby” problems, designed only to help the students acquire the necessary schemata for solving more substantial problems. In this particular lesson, the warmups are to help the student automate the first steps of calculating a derivative: remembering the formula, substituting in the point, and putting in the function correctly. The remaining details of the calculations are deliberately trivial simplifications; this is not the place for challenging problems.

Students are not expected to do all of the warmups; instead they are told how well they should be able to do them to be sufficiently “warmed up” before moving on. It is anticipated that brighter students will do one or two warmups and less able students several more. There are deliberately more warmups than most students will want to complete, to bore even the most anxious student into moving on to more stimulating problems.

3. **Transfer**, addressed by a problems segment at the end of each lesson. We now want the students to apply what they’ve practiced in the warmup sections to “real” problems. This section consists of several worked examples each followed by a similar problem to be completed (with a link to the solution). The worked examples are annotated to point out subtleties in the calculation, suggestions for doing calculations more efficiently, etc.. In general, this section can include more challenging problems (e.g. Problem 3.1.3); alternately, challenging problems can be left for a more extensive problems section at the end of the current module.

Design and Technology Notes

- **File format** The lesson is in the format of a PDF file, currently the most effective way of producing onscreen mathematical text. PDF files have other useful features: the pages can be different sizes (allowing more appropriate chunking of content than print), and can contain multimedia elements such as links, sound, movies, etc..
- **CAS** The lesson is meant to be used with a CAS (a computer algebra system, such as [LiveMath](#)) capable of graphing and zooming in on graphs readily. The clickable icon  is used to indicate that a CAS is to be used, and can link either to an empty CAS notebook or one pre-configured with the example at hand.
- **Page design** The font used is Verdana 14, for its screen readability (the KILL rule - keep it large and legible). Devices such as bullets, headings, boxes, etc, are used to further chunk content (see C. Kilian, [Writing for the Web](#), Self-Counsel Press, 1999). “Flags” are used to chunk examples, etc. by marking their beginning and end; these were drawn in Illustrator and fused with an editable text box in AppleWorks 6. The rounded lesson and section headers were designed to reflect the other rounded page elements (speaker icons, flags, ...), and the pastel colours were chosen to enhance the content structure without competing with it for attention.
- **Technology used to create pages** The graphs were created in LiveMath and edited in Illustrator. The pages were assembled in AppleWorks 6, using [MathType](#) to create the mathematical text, and then converted to PDF files. Adobe Acrobat was used to add links and sound files, which were recorded in [Sound Studio](#).
- **File size** The main drawback to using voiceovers this way is file size: this present file is close to 7 meg., and would make a prohibitively slow download for an online course (though byteserving the file would make for a shorter “subjective” download time). As part of a course delivered on CD-ROM, it would work fine (based on an estimate of 40 lessons of about the same size), with room left over for supplemental material.

Sweller's Instructional Design Principles (somewhat interpreted in the context of this lesson)

Use goal-free problems

Construct problems that encourage students to work forward from the givens towards an unspecified goal. Example: "find the derivative of ..." is goal-free, while "show that the derivative of ... is ..." is not. Problems that encourage students to work back from a known goal to the givens require extra "means-end" analysis not contributing directly to learning. Discourage students from checking the answer to a problem for a "hint" before attempting a solution, as this will turn a goal-free problem into one with a goal.

Use worked examples

Increase the number of worked examples relative to the number of problems to be solved. Students learn concepts and techniques more readily by studying worked examples than by puzzling them out through problem-solving. Students peeking at the answers in the back of the book for hints may be indicating a need for more worked examples.

Avoid split attention

Integrate any textual information necessary for understanding graphical materials directly into those materials. Instruction or information about a graph or diagram that needs to be processed to understand it should appear directly on the graph or diagram. If textual material necessary for understanding is physically separate from the diagram, the student is forced to integrate them mentally, increasing the working memory load unnecessarily.

Avoid redundancy

Avoid simultaneous presentation of the same material in different formats (e.g. as diagram labels and as separate text). Reading redundant material is an extraneous load on working memory; the material cannot be ignored, since it cannot be seen to be redundant until after it has been read. Eliminate the extraneous material.

Use audio

Since visual and aural working memory appear to be separate, using both can increase total working memory. Audio should be restricted to non-technical parts of the material to be learned (motivation, intuitive explanations, instructions, ...). To avoid redundancy, it should not merely reproduce printed text: talk *about* the text or diagram. If possible, audio should be student-controlled, to be listened to when needed and as often as needed.